Derivative of composite and implicit functions

If $z = y \cdot \sin(x - y)$, where $x = f(t)$ and $y = g(t)$ are defined by the system:

$$\begin{cases} x^2 + 2xy = t \\ 3x^2 + 7y^2 = 3t \end{cases}$$

1. Express $d^2z(x, y)$.
2. Find $\frac{dz}{dt}$.

Solution

1. We calculate the first-order partial derivatives of z:

   - $\dfrac{\partial z}{\partial x} = y\cos(x-y)$
   - $\dfrac{\partial z}{\partial y} = \sin(x-y) - y\cos(x-y)$

   Now, calculate the second-order partial derivatives:

   - $\dfrac{\partial^2 z}{\partial x^2} = -y\sin(x-y)$
   - $\dfrac{\partial^2 z}{\partial x \partial y} = \cos(x-y) + y\sin(x-y)$
   - $\dfrac{\partial^2 z}{\partial y^2} = -2\cos(x-y) + y\sin(x-y)$

   The second-order differential is:

   $$d^2 z = \frac{\partial^2 z}{\partial x^2}(dx)^2 + 2\frac{\partial^2 z}{\partial x \partial y}dx\,dy + \frac{\partial^2 z}{\partial y^2}(dy)^2$$

   Substituting the second-order partial derivatives:

   $$\begin{aligned} d^2 z = {} & \left(-y\sin(x-y)\right)(dx)^2 + 2\left(\cos(x-y) + y\sin(x-y)\right)dx\,dy \\ & + \left(-2\cos(x-y) + y\sin(x-y)\right)(dy)^2 \end{aligned}$$

2. Using the chain rule:
   $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

   The partial derivatives of z are:

   - $\dfrac{\partial z}{\partial x} = y\cos(x-y)$
   - $\dfrac{\partial z}{\partial y} = \sin(x-y) - y\cos(x-y)$

   Consider the implicit functions:

   $$\begin{cases} F_1(x,y,t) = x^2 + 2xy - t = 0 \\ F_2(x,y,t) = 3x^2 + 7y^2 - 3t = 0 \end{cases}$$

   We calculate the partial derivatives:

   - $\dfrac{\partial F_1}{\partial x} = 2x + 2y$
   - $\dfrac{\partial F_1}{\partial y} = 2x$
   - $\dfrac{\partial F_1}{\partial t} = -1$
   - $\dfrac{\partial F_2}{\partial x} = 6x$

- $\dfrac{\partial F_2}{\partial y} = 14y$
- $\dfrac{\partial F_2}{\partial t} = -3$

The system of equations can be expressed as:

$$\begin{cases} (2x+2y)\dfrac{dx}{dt} + 2x\dfrac{dy}{dt} = 1 \quad (1) \\ \\ 6x\dfrac{dx}{dt} + 14y\dfrac{dy}{dt} = 3 \quad (2) \end{cases}$$

In matrix form:

$$\begin{pmatrix} 2x+2y & 2x \\ 6x & 14y \end{pmatrix} \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Denote the Jacobian J:

$$J = \begin{pmatrix} 2x+2y & 2x \\ 6x & 14y \end{pmatrix}$$

Compute the determinant of the Jacobian:

$$\det J = (2x+2y)(14y) - (2x)(6x) = 28xy + 28y^2 - 12x^2$$

Invert the Jacobian:

$$J^{-1} = \frac{1}{\det J} \begin{pmatrix} 14y & -2x \\ -6x & 2x+2y \end{pmatrix}$$

Calculate $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$:

$$\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = J^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{\det J} \begin{pmatrix} 14y & -2x \\ -6x & 2x+2y \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Perform the matrix multiplication:

- $\dfrac{dx}{dt} = \dfrac{1}{\det J}(14y \cdot 1 + (-2x) \cdot 3) = \dfrac{14y - 6x}{\det J}$
- $\dfrac{dy}{dt} = \dfrac{1}{\det J}(-6x \cdot 1 + (2x+2y) \cdot 3) = \dfrac{-6x + 6x + 6y}{\det J} = \dfrac{6y}{\det J}$

Using the chain rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

The partial derivatives of z are:

- $\dfrac{\partial z}{\partial x} = y\cos(x-y)$
- $\dfrac{\partial z}{\partial y} = \sin(x-y) - y\cos(x-y)$

Substitute $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$:

$$\frac{dz}{dt} = y\cos(x-y)\left(\frac{14y-6x}{\det J}\right) + (\sin(x-y) - y\cos(x-y))\left(\frac{6y}{\det J}\right)$$